Assignment 12

1. Consider the function
$$f(x) = \frac{1}{2}x + x^2 \sin\frac{1}{x}, \quad x \neq 0,$$
and set $f(0) = 0$. Show that f is differentiable at 0 with $f'(0) = 1/2$ but it has no local inverse at 0. Does it contradict the inverse function theorem?

2. Find the partial derivatives of the inverse function at the designated points:

   (a) $F(u, v) = (u^2 + 5uv + v^2, u^2 - v^2)$, at $(3, 0)$.

   (b) $G(x, y, z) = (xy + z, z\sin\pi x, 6xy^2 - 5)$, at $(1, 0, -2)$.

3. (a) Consider the function
$$h(x, y) = (x - y^2)(x - 3y^2), \quad (x, y) \in \mathbb{R}^2.$$
Show that the set $\{(x, y) : h(x, y) = 0\}$ cannot be expressed as a local graph of a C^1-function over the x or y-axis near the origin.

   (b) Consider
$$\varphi(x, y) = (x - y^2)(-x + 3y^2), \quad (x, y) \in \mathbb{R}^2.$$
Show that the set $\{(x, y) : \varphi(x, y) = 0\}$ can be expressed as the local graphs of two C^1-functions over the y-axis near the origin in two different ways.

4. **(Revised)** Consider the system
$$2x^2 + y^2 - z^2 + w^2 = 4, \quad xyw - xyz = 1.$$
Show that x and w can be expressed as functions of y, z, that is, $x = f(y, z), w = g(y, z)$ near $(x, y, z, w) = (1, 1, 0, 1)$ and find the partial derivatives of f and g at this point.

5. Consider the system
$$x^2 - y^2 - u^3 + v^2 + 4 = 0, \quad 2xy + y^2 - 2u^2 + 3v^4 + 8 = 0$$
at $(2, -1, 2, 1)$. Show that there exist an open set U containing $(2, -1)$ and an open set W containing $(2, 1)$ and function $F = (f_1, f_2)$ from U to W, $F(2, -1) = (2, 1)$, such that
$$x^2 - y^2 - f_1(x, y)^3 + f_2(x, y)^2 + 4 = 0, \quad 2xy + y^2 - 2f_1(x, y)^2 + 3f_2(x, y)^4 + 8 = 0$$
hold. Find $\partial u/\partial x$ and $\partial^2 u/\partial x^2$ in terms of x, y, u and v.

In the following we are concerned with the inverse function theorem in infinite dimensional Banach spaces. On $C^k[a, b]$ the norm
$$\|f\|_{C^k} = \|f\|_\infty + \cdots + \|f^{(k)}\|_\infty$$
is given. It is easily seen that it makes $C^k[a, b]$ a Banach space. The first two problems are about the norm of a linear operator.

6. Optional. A linear operator (map) $T : X \to Y$ between two normed spaces X, Y is *bounded* if there exists some constant M such that
$$\|Tx\|_Y \leq M\|x\|_X, \quad \forall x \in X.$$

(a) Show that

$$\sup\left\{\frac{\|Tx\|_Y}{\|x\|_X},\ x \neq 0 \in X\right\} = \inf\left\{M:\ \|Tx\|_Y \leq M\|x\|_X\right\},$$

(b)

$$\|T\|_{op} = \sup\left\{\frac{\|Tx\|_Y}{\|x\|_X},\ x \neq 0 \in X\right\}$$

is norm on $L(X,Y)$, the vector space of all bounded, linear operators from X to Y. It is called the operator of T.

7. Optional. Consider the norms $\|\cdot\|_1$ and $\|\cdot\|_2$ define on $\mathbb{R}^n$. Show that the operator norms of the linear operator associated to the $n \times n$-matrix $\{a_{ij}\}$ with respect to $\|\cdot\|_1$ and $\|\cdot\|_2$ are given respectively by

$$\max_j \sum_{i=1}^n |a_{ij}|, \qquad \text{and } \max\{|\lambda_1|, \cdots, |\lambda_n|\}.$$

Recall that $\|x\|_1 = \sum_j |x_j|$ and $\|x\|_2 = (\sum_j x_j^2)^{1/2}$. This problem is related to Lemma 4.2.

8. Optional. Let $F: U \subset X \to Y$ where U is open set in X and X, Y are normed spaces. F is called *differentiable* at $x \in U$ if there exists a bounded linear operator $T: X \to Y$ such that

$$\lim_{\varepsilon \to 0} \left\| \frac{F(x+\varepsilon z) - F(x)}{\varepsilon} - Tz \right\|_Y = 0, \quad \forall z \in X.$$

This operator T is called the *derivative* of F at x and is denoted by $F'(x)$. F is in $C^1(U)$ if it is differentiable at every point of U and $(x,z) \mapsto F'(x)z$ is continuous from $U \times X$ to Y. Find the derivatives in the following cases and verify that these maps are C^1.

(a)

$$F(y) = y'(x) - \cos y(x), \quad X = \{y \in C^1[0,1]:\ y(0) = 0\},\ Y = C[0,1].$$

(b)

$$G(y) = y''(x) + y^3(x) - 1, \quad X = \{y \in C^2[0,1]:\ y(0) = y'(0) = 0\},\ Y = C[0,1].$$

(c)

$$I(y) = \int_{-\pi}^{\pi} \left(y'^2(x) + e^{y(x)}\right)dx, \quad X = C[-\pi,\pi],\ Y = \mathbb{R}.$$

9. Optional. A bounded linear operator from X to Y is called *invertible* if its inverse exists and is a bounded linear operator. Show that the inverse function theorem, Theorem 4.1, still holds when F is a C^1-map from $U \subset X$ to Y where both X and Y are Banach spaces. If too difficult, look up books on nonlinear functional analysis, for instance, chapter 4 in Deimling's "Nonlinear Functional Analysis" or chapter 3 in Berger's "Nonlinearity and Functional Analysis".

10. Optional. Consider the *boundary value problem* for the second order equation

$$y'' + a\sin y = f(x), \quad y(0) = y(\pi) = 0,\ x \in [0,\pi],$$

where $a \in (0,1)$ is fixed.

(a) Show that $F(y) = y'' + a\sin y$ is C^1 from $\{f \in C^2[0,\pi] :\ y(0) = y(\pi) = 0\}$ to $C[0,\pi]$.

(b) Show that its derivative at the function $y \equiv 0$ is given by $F'(0)z = z'' + az$ and it is invertible.

(c) Apply the inverse function theorem to show that

$$y'' + a\sin y = f(x), \quad y(0) = y(\pi) = 0,\ x \in [0,\pi],$$

is solvable for small f.